High order methods for the monodomain model

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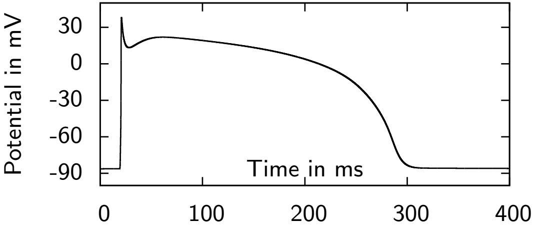
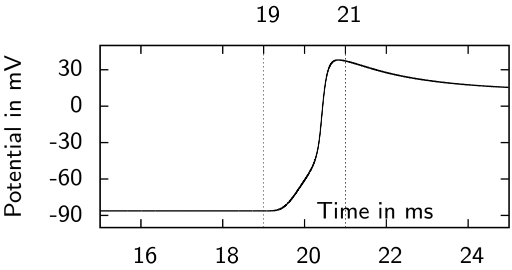
Discretization errors, why ?



Sharp depolarization

Discretization errors, why ?

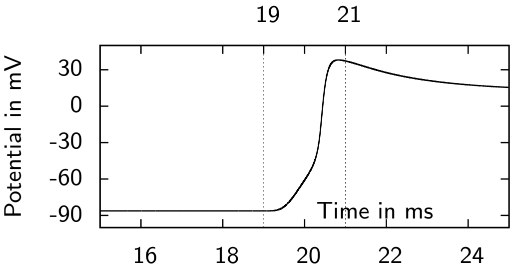


Sharp depolarization

- TNNP model:

  Depolarization duration $\simeq 1/500$ s

Discretization errors, why ?



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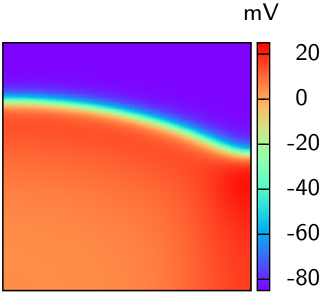
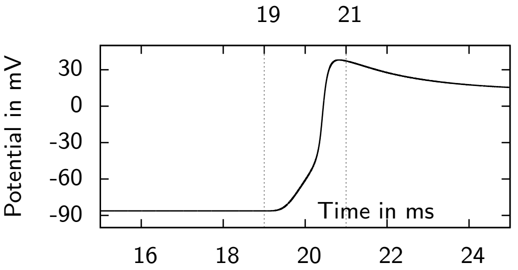


Sharp wavefronts

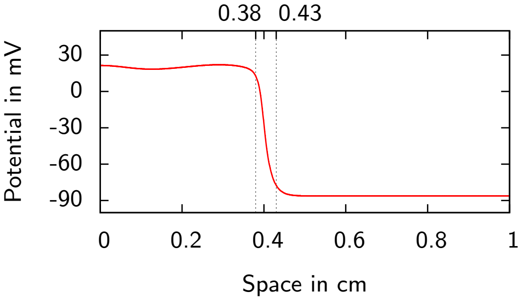
Discretization errors, why ?



Sharp depolarization

- TNNP model:

Depolarization duration
$\simeq 1/500$ s

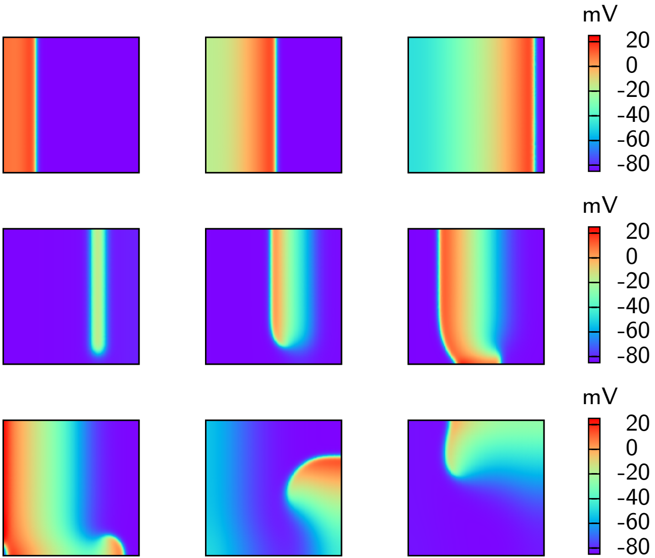


Sharp wavefronts

- TNNP model
- $A_m = 1000$ cm^{-1}

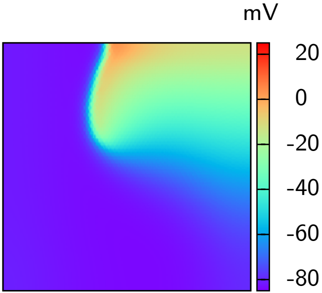
Wavefront thickness:
$\simeq 1/20$ cm

Spiral wave: description



Spiral wave: discretisation error

At a fixed time instant: $t = 160$ ms



Order 3 finite elements

- $\Delta x = 0.6$ mm

Order 3 time stepping-method

- $\Delta t = 0.1$ ms

Discretization error: **0.7 %**

Spiral wave: discretisation error

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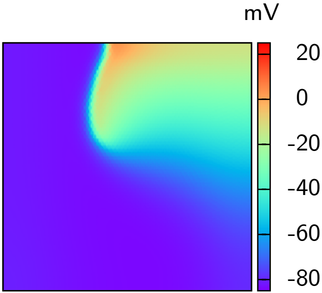

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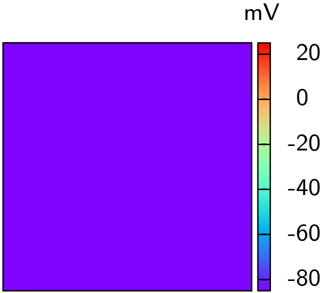


Order 1 finite elements

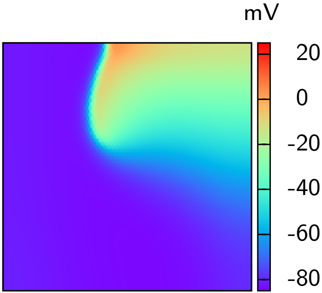
- $\Delta x = 0.6$ mm

Order 1 time stepping-method

- $\Delta t = 0.1$ ms

Spiral wave: discretisation error

At a fixed time instant: $t = 160$ ms



Order 3 finite elements

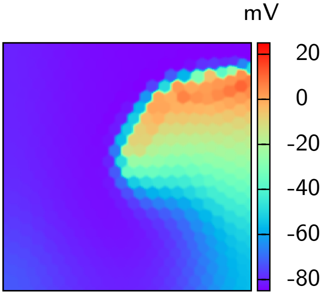
- $\Delta x = 0.6$ mm

Order 3 time stepping-method

- $\Delta t = 0.1$ ms

Discretization error: **0.7 %**

- CPU $\simeq 2.5$ s



Order 1 finite elements

- $\Delta x = 0.6$ mm

Order 1 time stepping-method

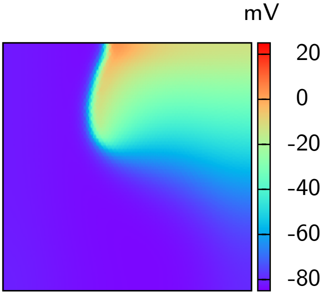
- $\Delta t = 0.1/16$ ms

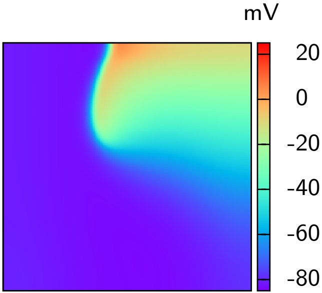
Discretization error: **40 %**

- CPU $\simeq 6.0$ s

Spiral wave: discretisation error

At a fixed time instant: $t = 160$ ms





Order 3 finite elements

- $\Delta x = 0.6$ mm

Order 3 time stepping-method

- $\Delta t = 0.1$ ms

Discretization error: **0.7 %**

- CPU $\simeq$ 2.5 s

Order 1 finite elements

- $\Delta x = 0.6/8$ mm $\simeq$ 75 µm

Order 1 time stepping-method

- $\Delta t = 0.1 \ / \ 16$ ms

Discretization error: **2.5 %**

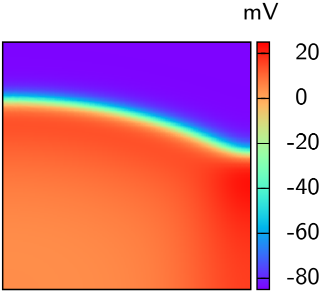
- CPU $\simeq$ 215 s

Outline

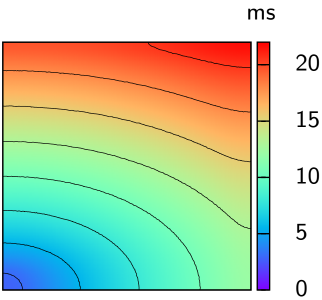
What error ?

- Mathematical errors
- *Physiological* errors : errors on activation times

  errors on the wavefront celerity



Potential at time $t=$ 16 ms



Activation times

Outline

What error ?

- Mathematical errors
- *Physiological* errors : errors on activation times

  errors on the wavefront celerity

Total discretization error:

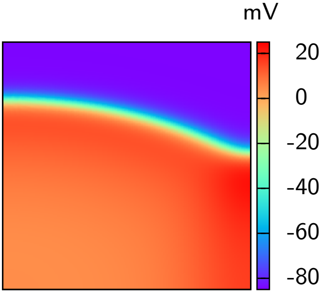
- total error $\leq$ (error in space) $+$ (error in time)

Outline:

1. Discretization error in space
2. Time-stepping methods

Test case

- square geometry (1 cm^2)
- constant anisotropy (horizontal fibres)
- Beeler-Reuter model
- $A_m = 500\ cm^{-1}$
- single-site stimulation (at the origin)



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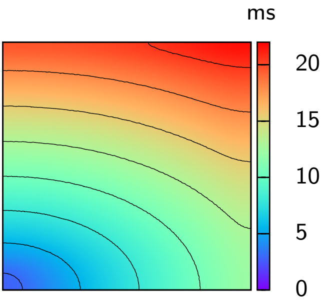
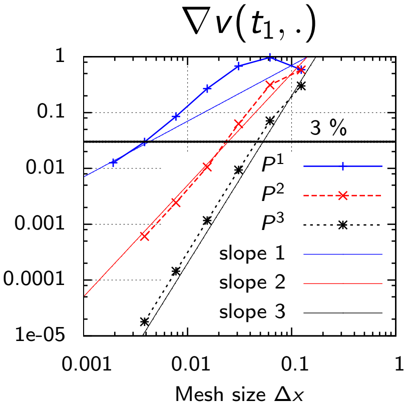


Activation times

Discretization error in space, 1

Relative error (in L^2-norm) on:



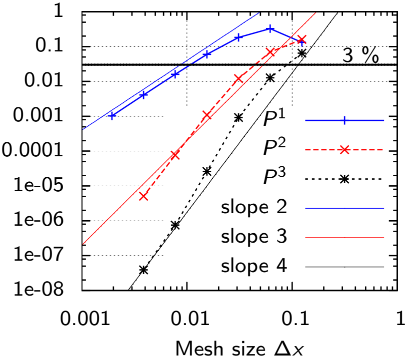
Error = $O(\Delta x^k)$

$v(t_1, .)$= transmembrane potential at time t_1= 16 ms

Discretization error in space, 1

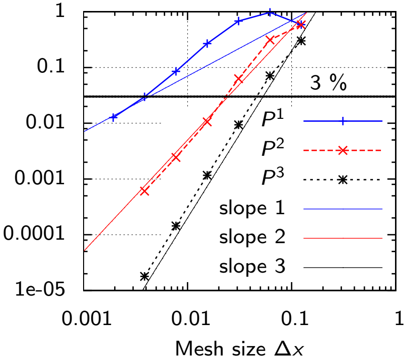
Relative error (in L^2-norm) on:

$v(t_1, .)$



Error $= O(\Delta x^{k+1})$

$\nabla v(t_1, .)$

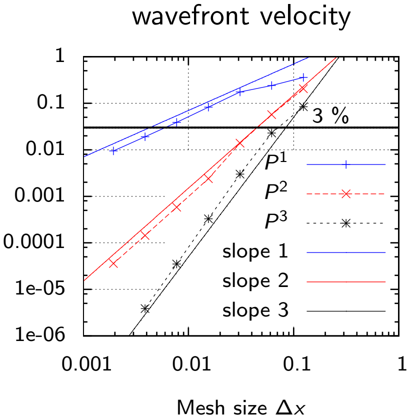


Error $= O(\Delta x^{k})$

$v(t_1, .)=$ transmembrane potential at time $t_1 = 16$ ms

Discretization error in space, 2

Relative error (in L^2-norm) on:

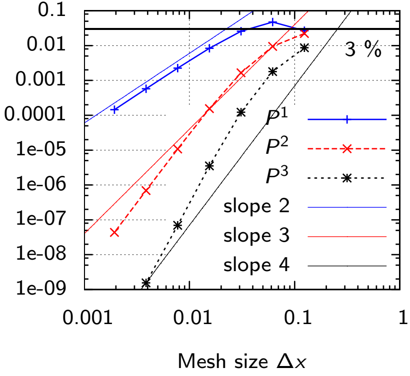


Error $= O(\Delta x^k)$

Discretization error in space, 2

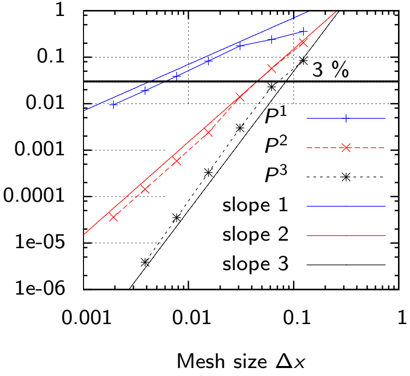
Relative error (in L^2-norm) on:

activation times



Error = $O(\Delta x^{k+1})$

wavefront velocity



Error = $O(\Delta x^{k})$

Discretization error in space, 3

Maximal mesh size

to have a discretization error in space

below 3%

Error type:	$v(t_1, .)$	$\nabla v(t_1, .)$	act. times	wave speed
P^1	0.01	4×10^{-3}	0.03	6×10^{-3}
P^2	0.04	0.02	0.1	0.04
P^3	0.09	0.04	0.2	0.07

Conclusion

- From order 1 to order 2: 4 time coarser mesh
- From order 1 to order 3: 8 time coarser mesh

Time-stepping

Equation for the gating variables w:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{w_\infty - w}{\tau} = a(v,w)w + b(v,w)$$

with $a(v,w) = -1/\tau$ and $b(v,w) = w_\infty/\tau$.

Time-stepping

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$$w_{n+1} = w_n + \Delta t \varphi_1(\alpha_n \Delta t)\Big(\alpha_n w_n + \beta_n\Big)$$

for $\varphi_1(z) = \big(\mathrm{e}^z - 1\big)/z$ with:

$$\alpha_n = a\big(v_n, w_n\big) := a_n$$

$$\beta_n = b\big(v_n, w_n\big) := b_n$$

Rush-Larsen (1978): order 1.

Time-stepping

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for $\varphi_1(z) = \big(\mathrm{e}^z - 1\big)/z$ with:

$$\alpha_n = \frac{3}{2}a_n - \frac{1}{2}a_{n-1}$$

$$\beta_n = \frac{3}{2}b_n - \frac{1}{2}b_{n-1}$$

Perego-Venezziani (2009): order 2.

Time-stepping

Equation for the gating variables w:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{w_\infty - w}{\tau} = a(v,w)w + b(v,w)$$

with $a(v,w) = -1/\tau$ and $b(v,w) = w_\infty/\tau$.

$$w_{n+1} = w_n + \Delta t \varphi_1(\alpha_n \Delta t)\Big(\alpha_n w_n + \beta_n\Big)$$

for $\varphi_1(z) = \big(\mathrm{e}^z - 1\big)/z$ with:

$$\alpha_n = \frac{1}{12}(23a_n - 16a_{n-1} + 5a_{n-2})$$

$$\beta_n = \frac{1}{12}(23b_n - 16b_{n-1} + 5b_{n-2}) + \frac{\Delta t}{12}(a_n b_{n-1} - a_{n-1} b_n)$$

Coudière-Lontsi-Pierre (2017): order 3.

Time-stepping

$$w_{n+1} = w_n + \Delta t \varphi_1(\alpha_n \Delta t)\Big(\alpha_n w_n + \beta_n\Big)$$

for $\varphi_1(z) = \big(e^z - 1\big)/z$ with:

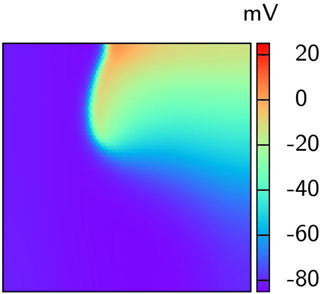
$$\alpha_n = \frac{1}{24}(55a_n - 59a_{n-1} + 37a_{n-2} - 9a_{n-3}),$$

$$\beta_n = \frac{1}{24}(55b_n - 59b_{n-1} + 37b_{n-2} - 9b_{n-3})$$
$$+ \frac{\Delta t}{12}(a_n(3b_{n-1} - b_{n-2}) - (3a_{n-1} - a_{n-2})b_n),$$

Coudière-Lontsi-Pierre (2017): order 4.

Spiral wave test case

- Total error in space = 4.3 %



- Beeler Reuter model (modified)
- $A_m = 600$ cm^{-1}
- Order 2 finite elements:
  $\Delta x = 0.6$ mm

Spiral wave test case

- Total error in space = 4.3 %

- Total discretization error with order k Rush-Larsen like scheme:

Order k:	1	2	3	4
$\Delta t = 0.1$ ms	–	18 %	**5.1** %	–
$\Delta t = 0.1/2$ ms	–	8.3 %	4.4 %	**4.2** %
$\Delta t = 0.1/4$ ms	23 %	**5.3** %	4.3 %	4.3 %
$\Delta t = 0.1/8$ ms	11.8 %	4.5 %	4.3 %	4.3 %

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$\Delta t = 0.1/4$ ms	23 %	**5.3** %	4.3 %	4.3 %
$\Delta t = 0.1/8$ ms	11.8 %	4.5 %	4.3 %	4.3 %

- Required CPU to have an error $\simeq 5$ %

Order k:	1	2	3	4
CPU	27	3.75	1	2.05

Future developments

More realistic test cases:

- complex domains with curved boundaries,
- 3D geometry.

Time stepping methods:

- improved stability,
- one-step methods.