Raviart-Thomas finite elements of Petrov-Galerkin type

François Dubois[1], Isabelle Greff[2], Charles Pierre[2]

[1] Laboratoire de Mathématiques, Université Paris-Sud, Orsay

[2] Laboratoire de Mathématiques et de leurs Applications de Pau, Université de Pau et des Pays de l'Adour.

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Motivation (1/4)

- We consider the Poisson problem

$$-\Delta u = f, \qquad u = 0 \quad \text{on} \quad \partial\Omega.$$

- discretized with mixed finite elements:

find $(u,p) \in \mathbb{P}^0 \times \mathrm{RT}^0$ so that for all $(v,q) \in \mathbb{P}^0 \times \mathrm{RT}^0$

$$-\left(\operatorname{div} p, v\right)_0 = \left(f, v\right)_0 , \qquad \left(p, q\right)_0 = -\left(u, \operatorname{div} q\right)_0$$

- The second equations says: $p = -\operatorname{div}^\star u$

$$\operatorname{div}^\star : \quad \mathbb{P}^0 \longrightarrow \mathrm{RT}^0$$

- **Non-local expression of the flux**

with respect to the scalar u.

Motivation (2/4)

- We want a local expression of the flux
- We skip to a Petrov-Galerkin mixed formulation:

find $(u, p) \in \mathbb{P}^0 \times \mathrm{RT}^0$ so that for all $(v, q) \in \mathbb{P}^0 \times X^\star$

$$-\left(\operatorname{div} p, v\right)_0 = \left(f, v\right)_0 \,, \qquad \left(p, q\right)_0 = -\left(u, \operatorname{div} q\right)_0 \qquad (1)$$

- Shape functions: scalar $\mathbb{P}^0$ || vector RT^0

$$\mathrm{RT}^0 = \operatorname{span}\left(\varphi_a,\ a \in \mathcal{F}\right) \qquad \text{classical basis}$$

- Test functions: scalar $\mathbb{P}^0$ || vector $X^\star \subset \mathrm{H}_{\mathrm{div}}$

$$X^\star = \operatorname{span}\left(\varphi_a^\star,\ a \in \mathcal{F}\right) \qquad \textbf{dual basis}$$

Motivation (3/4)

- Constraint 1. Orthogonality:

$$\left(\varphi_a^\star, \varphi_b\right)_0 = 0 \qquad \text{if} \qquad a \neq b \tag{C1}$$

- Constraint 2. Positivity:

$$\forall \quad a \in \mathcal{F}, \qquad \left(\varphi_a^\star, \varphi_a\right)_0 > 0 \tag{C2}$$

We can define: $\operatorname{div}^\star \; : \; \mathbb{P}^0 \longrightarrow X^\star$

$$\forall\ (u,p) \in \mathbb{P}^0 \times X, \quad \left(\operatorname{div} p, u\right)_0 = \left(p, \operatorname{div}^\star u\right)_0 .$$

- Definition 1. $\Pi \; : \; \mathrm{RT}^0 \longrightarrow X^\star, \qquad \Pi\, \varphi_a = \varphi_a^\star$
- Definition 2. Discrete gradient $\nabla_{\mathcal{T}} \; : \; \mathbb{P}^0 \longrightarrow \mathrm{RT}^0$

$$\begin{tikzcd} \mathrm{RT}^0 \arrow[r, "\operatorname{div}"] \arrow[d, "\Pi"'] & \mathbb{P}^0 \arrow[dl, "\operatorname{div}^\star"] \\ X^\star & \end{tikzcd} \qquad \nabla_{\mathcal{T}} := -\Pi^{-1} \operatorname{div}^\star$$

Motivation (4/4)

- Constraint 3. Flux equality:

$$\forall \quad a, \quad b \in \mathcal{F}, \qquad \int_b \varphi_a \cdot n_b \, \mathrm{d}s = \int_b \varphi_a^\star \cdot n_b \, \mathrm{d}s \qquad \text{(C3)}$$

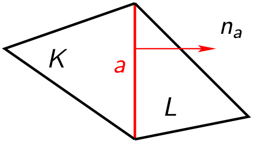
Proposition. Assume (C1), (C2) and (C3).

- The Petrov-Galerkin mixed formulation (1) is well-posed.
- The solution (u, p) satisfies

$$p = \nabla_{\mathcal{T}} u$$

- We have a local expression of the flux:

$$p = \nabla_{\mathcal{T}} u = \sum_{a \in \mathcal{F}} p_a \, \varphi_a \quad \Rightarrow \quad p_a = \frac{1}{\left(\varphi_a, \varphi_a^\star\right)_0} (u_L - u_K).$$



First results

[1] F. Dubois, 2000.

One dimensional case

- Stability condition $\Rightarrow$ $\left(\varphi_a, \varphi_a^\star\right)_0$ has a prescribed value.
- Discrete problem $\Leftrightarrow$ finite volumes.

[2] F. Dubois, I. Greff and C. Pierre, 2017.

Two dimensional case

- Introduction of a class of dual functions $\varphi_a^\star$
- $\left(\varphi_a, \varphi_a^\star\right)_0$ has a prescribed value
- Discrete problem $\Leftrightarrow$ 4 point finite volume scheme (R. Herbin, 1995).
- Stability is satisfied ($+$ error estimates).

General framework (1/3)

Relatively to a mesh $\mathcal{T}$ of the domain Ω:

- Shape functions $= \begin{cases} \text{scalar} & : \quad M \subset \mathrm{L}^2(\Omega) \\ \text{vector} & : \quad X \subset \mathrm{H}_{\text{div}}(\Omega) \end{cases}$

  and $\quad \text{div} \; : \quad X \longrightarrow M, \qquad \text{div}(X) = M.$

- Test functions $= \begin{cases} \text{scalar} & : \quad M \\ \text{vector} & : \quad X^\star \subset \mathrm{H}_{\text{div}}(\Omega) \end{cases}$

- Scalar basis functions: $\quad M = \text{span}\left(\chi_i, \; i \in \mathcal{C}\right)$

  Orthogonality : $\quad \left(\chi_i, \chi_{i'}\right)_0 = 0 \quad \text{if} \quad i \neq i'$

- Vector basis functions: $\begin{cases} X = \text{span}\left(\varphi_j, \; j \in \mathcal{F}\right) \\ X^\star = \text{span}\left(\varphi_j^\star, \; j \in \mathcal{F}\right) \end{cases}$

General framework (2/4)

- Constraint 1. Orthogonality:

$$\left(\varphi_j^\star, \varphi_{j'}\right)_0 = 0 \qquad \text{if} \qquad j \neq j' \tag{C1}$$

- Constraint 2. Positivity:

$$\forall \quad j \in \mathcal{F}, \qquad \left(\varphi_j^\star, \varphi_j\right)_0 > 0 \tag{C2}$$

We can define: $\operatorname{div}^\star \ : \quad M \longrightarrow X^\star$

$$\forall \ (u, p) \in M \times X, \quad \left(\operatorname{div} p, u\right)_0 = \left(p, \operatorname{div}^\star u\right)_0 .$$

- Definition 1. $\Pi \ : \quad X \longrightarrow X^\star, \qquad \Pi\, \varphi_a = \varphi_a^\star$
- Definition 2. discrete gradient $\nabla_{\mathcal{T}} \ : \quad M \longrightarrow X$

$$\begin{tikzcd} X \arrow[r, "\operatorname{div}"] \arrow[d, "\Pi"'] & M \arrow[dl, "\operatorname{div}^\star"] \\ X^\star & \end{tikzcd} \qquad \nabla_{\mathcal{T}} := -\Pi^{-1} \operatorname{div}^\star$$

General framework (3/4)

- Constraint 3.

$$\forall\ \varphi \in X\ , \qquad \operatorname{div}(\Pi\varphi - \varphi) \in M^{\perp} \qquad \text{(C3)}$$

- (C3) generalizes the flux equality constraint
- (C3) is a necessary condition for the algebraic relation

$$\forall\ (u, q) \in M \times X^{\star}, \quad \big(\operatorname{div} q, u\big)_0 = -\big(q, \nabla_{\mathcal{T}}\, u\big)_0$$

Proof.
$$\begin{aligned} -\big(q, \nabla_{\mathcal{T}}\, u\big)_0 &= \big(q, \Pi^{-1}\operatorname{div}^{\star} u\big)_0 && \leftarrow \text{def. } \nabla_{\mathcal{T}} \\ &= \big(\Pi^{-1}q, \operatorname{div}^{\star} u\big)_0 && \leftarrow \text{conseq. (C1)} \\ &= \big(\operatorname{div}\Pi^{-1}q, u\big)_0 && \leftarrow \text{def. } \operatorname{div}^{\star} \\ &= \big(\operatorname{div} q, u\big)_0 && \leftarrow \text{(C3)} \end{aligned}$$

General framework (4/4)

Proposition. Assume (C1), (C2) and (C3).

- The Petrov-Galerkin mixed formulation (1) is well-posed.
- The solution (u, p) satisfies

$$-\operatorname{div} \nabla_{\mathcal{T}} u = f_{\mathcal{T}}, \qquad p = \nabla_{\mathcal{T}} u$$

where $f_{\mathcal{T}}$ is the orthogonal projection of f on M.

- Local expression of the flux. Let $u \in M$, $\quad u = \sum_{i \in \mathcal{C}} u_i \, \chi_i$

and $\qquad p := \nabla_{\mathcal{T}} u \in X, \qquad p = \sum_{j \in \mathcal{F}} p_j \, \varphi_j$

Then $\qquad p_j = -\dfrac{1}{\left(\varphi_j, \varphi_j^\star\right)_0} \sum_{i \in \mathcal{C}} u_i \left(\chi_i, \operatorname{div} \varphi_j\right)_0$

Implementation

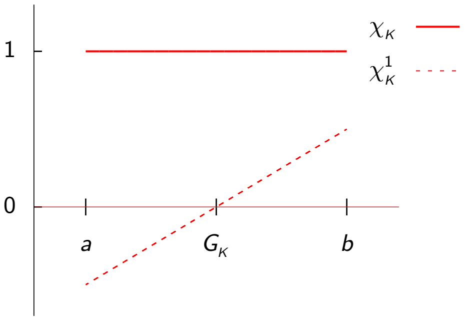
- **Notations:** $f_i = \left(\chi_i, f\right)_0, \qquad F = (f_i)_{i\in\mathcal{C}}$
$$u = \sum_{i\in\mathcal{C}} u_i\, \chi_i, \qquad U = (u_i)_{i\in\mathcal{C}}$$
- **The discrete problem:** find $u \in M$ such that
$$\operatorname{div} \nabla_{\mathcal{T}}\, u = f_{\mathcal{T}},$$
yields in matrix form: $\quad S\,U \;=\; F.$
- The matrix $S = [s_{ik}]_{i,\,k\in\mathcal{C}}$ is the matrix of the product
$$(u, v) \longrightarrow \left(\Pi\, \nabla_{\mathcal{T}}\, u, \nabla_{\mathcal{T}}\, v\right)_0,$$
and is symmetric, positive definite.
- The matrix coefficients are
$$s_{ik} = \sum_{j\in\mathcal{F}} \frac{1}{\left(\varphi_j, \varphi_j^\star\right)_0} \left(\chi_i, \operatorname{div} \varphi_j\right)_0 \left(\chi_k, \operatorname{div} \varphi_j\right)_0.$$

1D example (1)

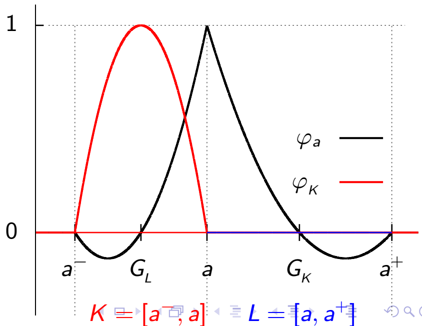
$$M = \mathbb{P}^1_d(\mathcal{T}) \quad \text{and} \quad X = \mathrm{RT}^1(\mathcal{T}) = \mathbb{P}^2(\mathcal{T})$$

- $\mathcal{T}$ is a mesh of an interval $\Omega \subset \mathbb{R}$.
- $\mathbb{P}^1_d(\mathcal{T})$ = piecewise affine (no continuity at the mesh vertices)
- $\mathbb{P}^2(\mathcal{T})$ = continuous + piecewise polynomial with degree ≤ 2

Scalar basis functions



Vector basis functions



1D example (2)

- Constraint. Support of the dual functions

$$\operatorname{supp} \varphi_j^\star \subset \operatorname{supp} \varphi_j. \tag{C4}$$

Proposition. Assume that (C1), (C3) and (C4) are satisfied:

- Then the scheme coefficients are

$$\left(\varphi_K^\star, \varphi_K\right)_0 = \frac{3}{2} |K|, \qquad \left(\varphi_a^\star, \varphi_a\right)_0 = \frac{1}{6} |\partial^c a|$$

with $\partial^c a$ the coboundary of the vertex a.

- The scheme is determined !

Computation of $\left(\varphi_K, \varphi_K^\star\right)_0$

$K = [a, b]$ is a mesh interval.

- Let $u \in M$ satisfying $u' = 1$. With (C3):

$$0 = \int_\Omega u \ \mathrm{div}(\varphi_K - \varphi_K^\star)\mathrm{d}x = \int_K u \ \mathrm{div}(\varphi_K - \varphi_K^\star)\mathrm{d}x$$
$$- \int_K u' \ (\varphi_K - \varphi_K^\star)\mathrm{d}x + [u \ (\varphi_K - \varphi_K^\star)]_a^b = \int_K (\varphi_K^\star - \varphi_K)\mathrm{d}x = 0$$

- On K: $\varphi_K + \varphi_a + \varphi_b = 1$

$$\int_\Omega \varphi_K^\star \varphi_K \, \mathrm{d}x = \int_K \varphi_K^\star \varphi_K \, \mathrm{d}x = \int_K \varphi_K^\star (1 - \varphi_a - \varphi_b)$$
$$= \int_K \varphi_K^\star \, \mathrm{d}x - \left(\varphi_K^\star, \varphi_a\right)_0 - \left(\varphi_K^\star, \varphi_b\right)_0 = \int_K \varphi_K^\star \, \mathrm{d}x$$

- Then $\left(\varphi_K, \varphi_K^\star\right)_0 = \int_K \varphi_K \, \mathrm{d}x$

Important remarks

- The dual functions $\varphi_j^\star$ never need to be computed in practice
- Choice of the basis for $M = \mathbb{P}_d^1(\mathcal{T})$:

  another basis $\Rightarrow$ equivalent scheme.

- Choice of the basis for $X = \mathrm{RT}^1(\mathcal{T})$:

  another basis $\Rightarrow$ different scheme.

Example:

if we impose $\int_K \varphi_K \,\mathrm{d}x = 1$ instead of $\varphi_K(G_K) = 1$

then $\left(\varphi_K^\star, \varphi_K\right)_0 = 0$

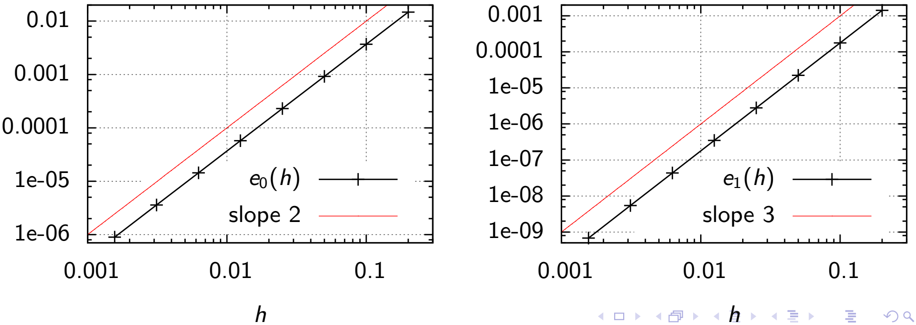
The positivity constraint (C2) is violated.

The scheme is ill-posed

Numerical test

- $-\Delta u = f$ with $u = \sin(\pi x)$ on $\Omega = (0,1)$
- Regular mesh $\mathcal{T}$ with size h
- Numerical solution: u_h
- Convergence in L^2norm

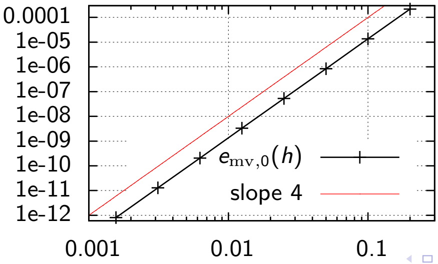
$$e_0(h) = \|u - u_h\|_0 \qquad e_1(h) = \|\nabla u - \nabla_{\mathcal{T}} u_h\|_0$$



Numerical test

- $-\Delta u = f$ with $u = \sin(\pi x)$ on $\Omega = (0,1)$
- Regular mesh $\mathcal{T}$ with size h
- Numerical solution: u_h
- Convergence of the mean values for u_h

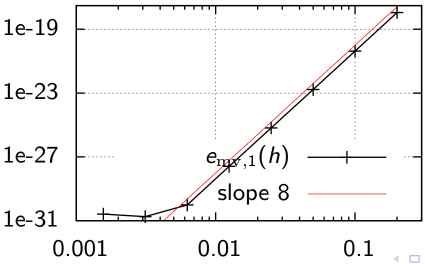
$$e_{\mathrm{mv},0}(h)^2 = \sum_{i\in\mathcal{C}} \left(\frac{1}{|K_i|} \int_{K_i} (u(x) - u_h(x))\mathrm{d}x \right)^2 |K_i|$$



Numerical test

- $-\Delta u = f$ with $u = \sin(\pi x)$ on $\Omega = (0,1)$
- Regular mesh $\mathcal{T}$ with size h
- Numerical solution: u_h
- Convergence of the mean values for $\nabla_{\mathcal{T}} u_h$

$$e_{\mathrm{mv},1}(h)^2 = \sum_{i \in \mathcal{C}} \left(\frac{1}{|K_i|} \int_{K_i} (\nabla u(x) - \nabla_{\mathcal{T}} u_h(x)) \mathrm{d}x \right)^2 |K_i|$$



Conclusion: work in progress in 2D

$$M = \mathbb{P}^1_d(\mathcal{T}) \quad \text{and} \quad X = \mathrm{RT}^1(\mathcal{T})$$

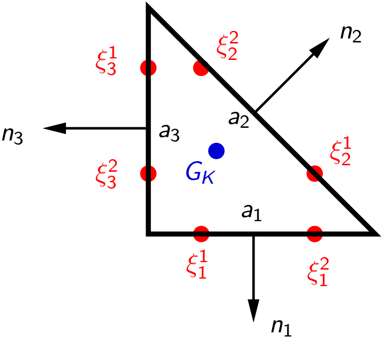
- $\mathcal{T}$ is a triangular mesh of the domain $\Omega \subset \mathbb{R}^2$.
- The classical basis of $\mathrm{RT}^1(\mathcal{T})$ cannot be considered
- We construct an alternative basis for $\mathrm{RT}^1(\mathcal{T})$:

$$\mathrm{RT}^1(\mathcal{T}) = \mathrm{span}\left(\varphi_a^j, \quad \varphi_K^j \right)$$

for j=1, 2,
for all edge a
for all triangle K

Conclusion: work in progress in 2D

For $\varphi \in \mathrm{RT}^1(K)$ on the reference triangle:



- G_K = centre of mass
- a_i = triangle edges
- n_i = unit normal to a_i
- ξ_i^1 and ξ_i^2 = Gaussian quadrature nodes on a_i

Degrees of freedom (6+2):

- Nodal values of the flux at the Gaussian nodes:

$$\frac{1}{|a_i|}\varphi(\xi_i^j)\cdot n_i$$

- Components of φ at the center of mass:

$$\varphi(G_K)\cdot\left|\begin{matrix}1\\0\end{matrix}\right.$$

$$\varphi(G_K)\cdot\left|\begin{matrix}0\\1\end{matrix}\right.$$

Conclusion: work in progress in 2D

- Constraint. Support of the dual functions

$$\operatorname{supp} \varphi_j^\star \subset \operatorname{supp} \varphi_j. \tag{C4}$$

Proposition. Assume that (C1), (C3) and (C4) are satisfied:

- Then the scheme coefficients are:
  for a triangle K

$$\left(\varphi_K^1, \varphi_K^{1\star}\right)_0 = \frac{3}{16}\frac{AB^2}{|K|}, \qquad \left(\varphi_K^2, \varphi_K^{2\star}\right)_0 = \frac{3}{16}\frac{AC^2}{|K|},$$

  for an edge a:

$$\left(\varphi_a^j, \varphi_a^{j\star}\right)_0 = \frac{1}{12}\frac{|\partial^c a|}{|a|^2},$$

- The scheme is determined !
- The computations are essentially similar to the 1D case.