Raviart Thomas Petrov-Galerkin Finite Elements

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Abstract

The general theory of Babuška ensures necessary and sufficient conditions for a mixed problem in classical or Petrov-Galerkin form to be well posed in the sense of Hadamard. Moreover, the mixed method of Raviart-Thomas with low-level elements can be interpreted as a finite volume method with a non-local gradient. In this contribution, we propose a variant of type Petrov-Galerkin to ensure a local computation of the gradient at the interfaces of the elements. The in-depth study of stability leads to a specific choice of the test functions. With this choice, we show on the one hand that the mixed Petrov-Galerkin obtained is identical to the finite volumes scheme "volumes finis à 4 points" ("VF4") of Faille, Gallouët and Herbin and to the condensation of mass approach developed by Baranger, Maitre and Oudin. On the other hand, we show the stability via an inf-sup condition and finally the convergence with the usual methods of mixed finite elements.

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1) Introduction

Discrete gradient

In the sequel, $\Omega \subset \mathbb{R}^2$ denotes an open bounded convex with a polygonal boundary. The functional spaces $\mathrm{L}^2(\Omega)$, $\mathrm{H}^1_0(\Omega)$ and $\mathrm{H}(\mathrm{div},\Omega)$ are considered. The L^2-scalar products on $\mathrm{L}^2(\Omega)$ and on $\left[\mathrm{L}^2(\Omega)\right]^2$ are similarly denoted $(\cdot,\cdot)_0$, without ambiguity. Being set a triangulation $\mathscr{T}$ of Ω, P^0 and RT denote the associated finite element spaces of the piecewise constant functions on the mesh and the Raviart Thomas vector fields of order 0 [10], precise definitions follow in Section 2.
The two unbounded operators, gradient

$$\nabla: \ \mathrm{L}^2(\Omega) \supset \mathrm{H}^1_0(\Omega) \to \left[\mathrm{L}^2(\Omega)\right]^2$$

and divergence

$$\mathrm{div}: \ \left[\mathrm{L}^2(\Omega)\right]^2 \supset \mathrm{H}(\mathrm{div},\Omega) \to \mathrm{L}^2(\Omega)$$

together satisfy the Green formula: for $u \in \mathrm{H}^1_0(\Omega)$ and $p \in \mathrm{H}(\mathrm{div},\Omega)$:

$$(\nabla u, p)_0 = -(u, \mathrm{div}\, p)_0$$

Identifying $\mathrm{L}^2(\Omega)$ and $\left[\mathrm{L}^2(\Omega)\right]^2$ with their topological dual spaces using the L^2-scalar product yields the following property,

$$\nabla = -\,\mathrm{div}^\star,$$

that is a weak definition of the gradient on $\mathrm{H}^1_0(\Omega)$.
We search to define a *discrete gradient* denoted $\nabla_{\mathscr{T}}$ on P^0 also based on a similar weak formalism. Starting from the divergence operator

$$\mathrm{div}: \ RT \to P^0,$$

one can define $\mathrm{div}^\star: \ (P^o)' \to (RT)'$, between the algebraic dual spaces of P^0 and RT respectively. The natural basis for P^0 is made of the indicator functions of the mesh triangles, that is orthogonal for the L^2-scalar product. Therefore, P^0 is identified with its algebraic dual space $(P^o)'$. On the contrary, the Raviart Thomas basis $\left\{\varphi_a,\, a \in \mathscr{T}^1\right\}$ of RT (denoting by $\mathscr{T}^1$ the mesh edge set, see Section 2) has no orthogonality property and cannot be used directly (see below) to identify RT with $(RT)'$. For this reason, a general identification process of $(RT)'$ to a subspace $RT^\star \subset \mathrm{H}(\mathrm{div},\Omega)$ so that,

$$RT^\star = \mathrm{Span}\left(\varphi^\star_a, \quad a \in \mathscr{T}^1\right),$$

with,

(1) $$\varphi^\star_a \in \mathrm{H}(\mathrm{div},\Omega), \quad (\varphi^\star_a, \varphi_a)_0 \neq 0,$$

and the orthogonality property,

(2) $$(\varphi^\star_a, \varphi_b)_0 = 0 \quad \text{for} \quad a, b \in \mathscr{T}^1, \quad a \neq b,$$

is considered. Setting $\Pi: \ RT \to RT^\star$ with $\Pi\varphi_a = \varphi^\star_a$, we have the following diagram, and general definition for the discrete gradient,

(3) $$\begin{array}{ccc} RT & \xrightarrow{\mathrm{div}} & P_0 \\ \Pi\big\downarrow & & \big\downarrow id \\ RT^\star & \xleftarrow[\mathrm{div}^\star]{} & P_0 \end{array}, \qquad \nabla_{\mathscr{T}} = -\Pi^{-1}\,\mathrm{div}^\star \ : \ P^0 \to RT.$$

The definition of the discrete gradient is effective once $\{\varphi_a^\star, a \in \mathscr{T}^1\}$ has been set. Various choices are possible. The first choice is to set $RT^\star = RT$, and therefore to build $\{\varphi_a^\star, a \in \mathscr{T}^1\}$ with a Gram Schmidt orthogonalization process on the Raviart Thomas basis. Such a choice has an important drawback. The dual base function $\varphi_a^\star$ does not conserve a support located around the edge a. The discrete gradient matrix will be a full matrix related with the Raviart Thomas mass matrix inverse. This is not relevant with regard to the original gradient operator that is local in space. This choice corresponds to the classical mixed finite element discrete gradient that is known to be associated with a full matrix. In order to overcome this problem, Baranger, Maitre and Oudin [2] proposed to lump the mass matrix of the mixed finite element method. By doing this, they obtain a discrete *local* gradient.

A second choice, proposed in Thomas-Trujillo [11] and also by one of us in [3, 4, 5], that will be investigated in this paper, is to search for a dual basis satisfying, in addition to the orthogonality property (2), the localization constraint,

(4) $$\forall\, a \in \mathscr{T}^1, \quad \mathrm{Supp}(\varphi_a^\star) \subset \mathrm{Supp}(\varphi_a),$$

in order to impose locality to the discrete gradient. With such a constraint the discrete gradient of $u \in P^0$ will be defined on each edge $a \in \mathscr{T}^1$ only from the two values of u on each side of a. In this context it is no longer asked to have $\varphi_a^\star \in RT$ so that $RT \neq RT^\star$: thus, this is a Petrov-Galerkin discrete formalism, as defined and used *a priori* in the article of Babuška [1].

2) Background and notations

Meshes

A conformal triangle mesh $\mathscr{T}$ of Ω in the sense followed by is considered. The angle, vertex, edge and triangle sets of $\mathscr{T}$ are respectively denoted $\mathscr{T}^{-1}$, $\mathscr{T}^0$, $\mathscr{T}^1$ and $\mathscr{T}^2$. For $K \in \mathscr{T}^2$ (*resp.* $a \in \mathscr{T}^1$) its area (*resp.* length) is denoted $|K|$ (*resp.* $|a|$).

Let $K \in \mathscr{T}^2$. Its three edges are denoted $a_{K,i}$, the unit normal to $a_{K,i}$ pointing outwards K is denoted $n_{K,i}$, Its three vertices and angles are denoted $W_{K,i}$ and $\theta_{K,i}$ respectively, so that $W_{K,i}$ and $\theta_{K,i}$ are opposite to $a_{K,i}$ (see Fig. 1).

Let $a \in \mathscr{T}^1$. One of its two unit normal is chosen and denoted n_a. This sets an orientation for a. Let S_a, N_a be the two vertices of a, ordered so that (n_a, S_aN_a) has a direct orientation.

The sets $\mathscr{T}_i^1$ and $\mathscr{T}_b^1$ of the internal and boundary edges respectively are defined as,

$$\mathscr{T}_b^1 = \left\{a \in \mathscr{T}^1, \quad a \subset \partial\Omega\right\}, \; \mathscr{T}_i^1 = \mathscr{T}^1 - \mathscr{T}_b^1 .$$

Let $a \in \mathscr{T}_i^1$. Its coboundary $\partial^c a$ is made of a unique ordered pair $K, L \in \mathscr{T}^2$ so that $a \subset \partial K \cap \partial L$ and so that n_a points from K towards L. In such a case the following notation will be used:

$$a \in \mathscr{T}_i^1, \; \partial^c a = (K, L)$$

and we will denote W_a (*resp.* E_a) the vertex of K (*resp.* L) opposite to a. Let $a \in \mathscr{T}_b^1$, n_a is assumed to point towards the outside of Ω. Its coboundary is made of a single $K \in \mathscr{T}^2$ so that $a \subset \partial K$, which situation is denoted as follows:

$$a \in \mathscr{T}_b^1, \; \partial^c a = (K)$$

and we will denote W_a the vertex of K opposite to a. If $a \in \mathscr{T}^1$ is an edge of $K \in \mathscr{T}^2$, the angle of K opposite to a is denoted $\theta_{a,K}$ (see Fig. 2).

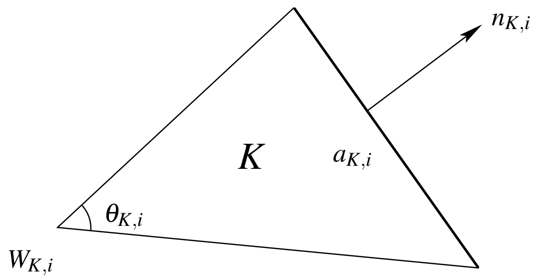

Fig. 1 Mesh notations for a triangle $K \in \mathscr{T}^2$

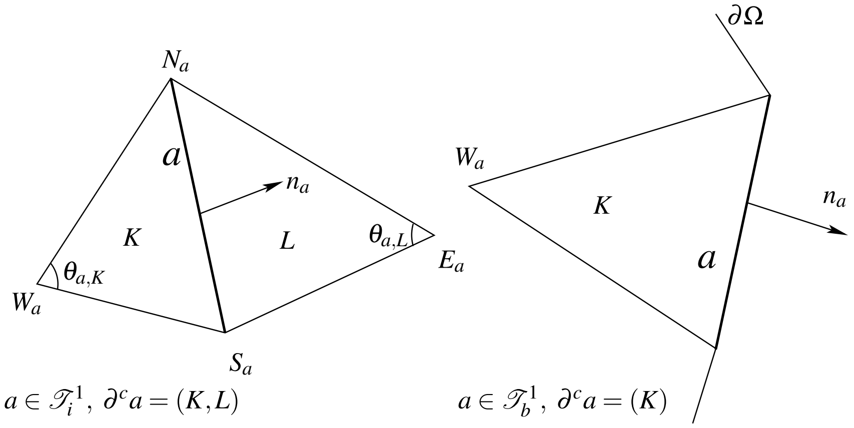

Fig. 2 Mesh notations for an internal edge (left) and for a boundary edge (right)

The finite element spaces

Relatively to a mesh $\mathscr{T}$, the finite element spaces P^0 and RT will be considered. The space $P^0 \subset \mathrm{L}^2(\Omega)$ is the space of piecewise constant functions on the mesh triangles. The indicators $\mathbb{1}_K$ for $K \in \mathscr{T}^2$ form a basis of P^0. To $u \in P^0$ is associated the vector $(u_K)_{K\in\mathscr{T}^2}$ so that

$$u = \sum_{K\in\mathscr{T}^2} u_K \, \mathbb{1}_K \, .$$

The space $RT \subset \mathrm{H}(\mathrm{div},\Omega)$ is the Raviart Thomas of order 0 finite element space introduced in [10]. An element $p \in RT$ is uniquely determined by its fluxes

$$p_a := \int_a p \cdot n_a \, \mathrm{d}s \ \text{ for } \ a \in \mathscr{T}^1 \, .$$

The classical basis $\{\varphi_a, \, a \in \mathscr{T}^1\}$ of RT is so that

$$\int_b \varphi_a \cdot n_b \, \mathrm{d}s = \delta_{ab} \ \text{ for all } b \in \mathscr{T}^1$$

and with δ_{ab} the Kronecker symbol. For each $p \in RT$ we can associate a discrete vector of fluxes $(p_a)_{a\in\mathscr{T}^1}$ and we have $p = \sum_{a\in\mathscr{T}^1} p_a \varphi_a$.

The *local Raviart Thomas basis functions* are defined, for $K \in \mathscr{T}^2$ and $i = 1, 2, 3$, by:

$$\varphi_{K,i}(x) = \frac{1}{2|K|} \nabla |x - W_{K,i}|^2 \quad \text{on} \quad K \quad \text{and} \quad \varphi_{K,i} = 0 \quad \text{otherwise.} \tag{5}$$

With that definition: $\varphi_a = \varphi_{K,i} - \varphi_{L,j}$ if $a \in \mathscr{T}_i^1$, $\partial^c a = (K,L)$ and $a = a_{K,i} = a_{L,j}$ $\varphi_a = \varphi_{K,i}$ if $a \in \mathscr{T}_b^1$, $\partial^c a = (K)$ and $a = a_{K,i}$ and it is retrieved that $\mathrm{Supp}(\varphi_a) = K \cup L$ if $a \in \mathscr{T}_i^1$, $\partial^c a = (K,L)$ or $\mathrm{Supp}(\varphi_a) = K$ in case $a \in \mathscr{T}_b^1$, $\partial^c a = (K)$. This provides a second way to decompose $p \in RT$ as,

$$p = \sum_{K \in \mathscr{T}^2} \sum_{i=1}^{3} p_{K,i}\, \varphi_{K,i}\,,$$

where $p_{K,i} = \varepsilon p_a$ if $a = a_{K,i}$ with $\varepsilon = n_a \cdot n_{K,i} = \pm 1$. Since $\operatorname{div} \varphi_{K,i} = \frac{1}{|K|}$, the divergence operator $\operatorname{div}: RT \to P^0$ is given by,

$$\operatorname{div} p = \sum_{K \in \mathscr{T}^2} (\operatorname{div} p)_K\, \mathbb{1}_K, \qquad (\operatorname{div} p)_K = \frac{1}{|K|} \sum_{1=1}^{3} p_{K,i}. \tag{6}$$

3) Raviart-Thomas dual basis

Definition 1

The family $(\varphi_a^\star)_{a \in \mathscr{T}^1}$ is said to be a Raviart Thomas dual basis if it satisfies (1), the orthogonality condition (2), the localization condition (4) and the following *flux normalization* condition:

$$\forall\, a,\ b \in \mathscr{T}^1, \quad \int_b \varphi_a^\star \cdot n_b \,\mathrm{d}s = \delta_{ab}, \tag{7}$$

as for the Raviart Thomas basis functions φ_a, see Section 2. In such a case, $RT^\star = \mathrm{Span}(\varphi_a^\star, a \in \mathscr{T}^1)$ is the associated Raviart Thomas dual space, $\Pi: \ \varphi_a \in RT \to \varphi_a^\star \in RT^\star$ the projection onto RT and $\nabla_{\mathscr{T}} = -\Pi^{-1} \operatorname{div}^\star: \ P^0 \to RT$ the associated discrete gradient, as described in diagram (3).

The following algebraic relations will be useful. From eq. (2) one can check that,

$$\forall\, p_1,\ p_2 \in RT, \quad (\Pi p_1, p_2)_0 = (p_1, \Pi p_2)_0\,. \tag{8}$$

The condition (7) implies with the divergence theorem that, $\forall\, p \in RT, \quad \forall\, K \in \mathscr{T}^2$, $\int_K \operatorname{div} p \,\mathrm{d}x = \int_K \operatorname{div}(\Pi p)\,\mathrm{d}x$, and so that,

$$\forall\, (u,p) \in P^0 \times RT, \quad (\operatorname{div} p, u)_0 = (\operatorname{div}(\Pi p), u)_0. \tag{9}$$

Now consider $u \in P^0$ and $q \in RT^\star$. We have with (9), $(u, \operatorname{div} q)_0 = (u, \operatorname{div}(\Pi^{-1} q))_0 = (\operatorname{div}^\star u, \Pi^{-1} q)_0$. Then with (8), $(u, \operatorname{div} q)_0 = (\Pi^{-1}(\operatorname{div}^\star u), q)_0$. As a result:

$$\forall\, u \in P^0, \quad \forall\, q \in RT^\star, \quad (u, \operatorname{div} q)_0 = -(\nabla_{\mathscr{T}} u, q)_0. \tag{10}$$

Proposition 1 [Computation of the discrete gradient]

Let $(\varphi_a^\star)_{a \in \mathscr{T}^1}$ be a Raviart Thomas dual basis. The discrete gradient is given for $u \in P^0$, by the relation $\nabla_{\mathscr{T}} u = \sum_{a \in \mathscr{T}^1} p_a \varphi_a$ with the conditions

$$\left\{ \begin{array}{rcl} \text{if } a \in \mathscr{T}_i^1,\ \partial^c a = (K,L),\ p_a & = & \dfrac{u_L - u_K}{(\varphi_a, \varphi_a^\star)_0} \\ \text{if } a \in \mathscr{T}_b^1,\ \partial^c a = (K),\ p_a & = & \dfrac{-u_K}{(\varphi_a, \varphi_a^\star)_0} \end{array} \right. . \tag{11}$$

This proposition deserves comments.

The result of the localization condition (4) is, as expected, a local discrete gradient: its value on an edge $a \in \mathscr{T}^1$ only depends on the values of the scalar function u on each sides of a. The definition of the discrete gradient on the external edges implicitly takes into account a zero value for the scalar data u on the domain boundary. This is relevant since the divergence with domain the full space $\mathrm{H}(\mathrm{div},\Omega)$ has for adjoint the gradient with domain $\mathrm{H}^1_0(\Omega)$, which adjoint property has been translated at a discrete level. The formulation of the discrete gradient in proposition 1 brings to the fore the coefficients $(\varphi_a^\star,\varphi_a)_0$: more details follow in the next subsection.

Petrov-Galerkin discretization for the Dirichlet Poisson problem
Consider the following Dirichlet Poisson problem on Ω,

$$-\Delta u = f \in \mathrm{L}^2(\Omega)\,,\ u = 0 \text{ on } \partial\Omega\,.$$

Consider a mesh $\mathscr{T}$ and a Raviart Thomas dual basis $(\varphi_a^\star)_{a\in\mathscr{T}^1}$. Let us denote $V = P^0 \times RT$ and $V^\star = P^0 \times RT^\star$. The mixed Petrov-Galerkin discretization of the Poisson problem is: find $(u,p) \in V$ so that,

$$\forall\ (v,q) \in V^\star,\quad (p,q)_0 + (u,\mathrm{div}\,q)_0 = 0 \quad \text{and} \quad -(\mathrm{div}\,p,v)_0 = (f,v)_0\,. \tag{12}$$

The mixed Petrov-Galerkin discrete problem (12) reformulates as: find $(u,p) \in V$ so that,

$$\forall\ (v,q) \in V^\star,\ B\big((u,p),(v,q)\big) = (f,v)_0$$

where the bilinear form B is defined for $(u,p) \in V$ and $(v,q) \in V^\star$ by,

$$B\big((u,p),(v,q)\big) = (u,\mathrm{div}\,q)_0 + (p,q)_0 - (\mathrm{div}\,p,v)_0\,.$$

Proposition 2 [Solution of the mixed discrete problem]
The pair $(u,p) \in V$ is a solution of problem (12) if and only if

$$\nabla_{\mathscr{T}}\, u = p, \qquad -\mathrm{div}(\nabla_{\mathscr{T}}\, u) = f_{\mathscr{T}}, \tag{13}$$

where $f_{\mathscr{T}} \in P^0$ is the projection of f, defined by,

$$f_{\mathscr{T}} = \sum_{K\in\mathscr{T}^2} f_K\, \mathbb{1}_K,\ f_K = \frac{1}{|K|}\int_K f\,\mathrm{d}x\,.$$

If $(\varphi_a,\varphi_a^\star) > 0$ for all $a \in \mathscr{T}^1$, then problem (12) has a unique solution.
Proposition 1 shows an equivalence between the mixed Petrov-Galerkin discrete problem (12) and the discrete problem (13). Problem (13) actually is a *finite volume like* problem. Precisely, it becomes: find $u \in P^0$ so that, for all $K \in \mathscr{T}^2$:

$$\sum_{\substack{a\in\mathscr{T}_i^1,\ \partial^c a=(K,L)\\ \text{or}\ \partial^c a=(L,K)}} \frac{u_L - u_K}{(\varphi_a^\star,\varphi_a)_0} + \sum_{a\in\mathscr{T}_b^1,\ \partial^c a=(K)} \frac{-u_K}{(\varphi_a^\star,\varphi_a)_0} = |K|\, f_K\,.$$

This *finite volume like* problem only involves the coefficient $(\varphi_a^\star,\varphi_a)_0$. We compute this scalar product in the next section.

4) Retrieving the "VF4" scheme

Let $g:\ (0,1) \to \mathbb{R}$ so that,

$$\int_0^1 g\,\mathrm{d}s = 1\,,\ \int_0^1 g(s)s^2\,\mathrm{d}s = 0 \text{ and } g(s) = g(1-s)\,.$$

On a mesh $\mathscr{T}$ are defined $g_{K,i}: \ a_{K,i} \to \mathbb{R}$ for $K \in \mathscr{T}^2$ and $i = 1, 2, 3$ as,

$$g_{K,i}(x) = \frac{g(s)}{|a_{K,i}|} \quad \text{for } x = s S_{K,i} + (1-s) N_{K,i}\,.$$

For $K \in \mathscr{T}^2$ is denoted $\delta_K \ : K \to \mathbb{R}$ a function that satisfies

$$\int_K \delta_K \, \mathrm{d}x = 1 \text{ and } \int_K \delta_K(x)\ |x - W_{K,i}|^2 \, \mathrm{d}x = 0 \text{ for } i = 1,\ 2,\ 3.$$

To a family $(\psi_{K,i})$ of functions on Ω for $K \in \mathscr{T}^2$ and for $i = 1, 2, 3$ is associated the family $(\psi_a)_{a \in \mathscr{T}^1}$ so that,

$$\left\{ \begin{array}{rcl} \text{if } a \in \mathscr{T}_i^1,\ \partial^c a = (K,L) \quad \text{and } a = a_{K,i} = a_{L,j}\,,\ \psi_a & = & \psi_{K,i} - \psi_{L,j} \\ \text{if } a \in \mathscr{T}_b^1,\ \partial^c a = (K) \text{ and } a = a_{K,i}\,,\ \psi_a & = & \psi_{K,i}\,. \end{array} \right. \tag{14}$$

Theorem 1 [Error estimations]
Assume that the mesh $\mathscr{T}$ angles all satisfy $0 < \theta_{K,i} < \pi/2$. Consider a family $(\varphi^\star_{K,i})$ of vector fields on Ω for $K \in \mathscr{T}^2$ and for $i = 1, 2, 3$ that satisfy, independently on i, on K,

$$\operatorname{div} \varphi^\star_{K,i} = \delta_K\,, \quad \varphi^\star_{K,i} = 0 \ \text{ otherwise} \tag{15}$$

and, on ∂K,

$$\varphi^\star_{K,i} \cdot n = g_{K,i} \text{ on } a_{K,i}\,, \quad \varphi^\star_{K,i} \cdot n = 0 \text{ otherwise}\,. \tag{16}$$

Let $(\varphi^\star_a)_{a \in \mathscr{T}^1}$ be constructed with eq. (14). Then $(\varphi^\star_a)_{a \in \mathscr{T}^1}$ is a Raviart Thomas dual basis. The coefficients $(\varphi^\star_a, \varphi_a)_0$ only depend on mesh $\mathscr{T}$ geometry, as follows

$$\left\{ \begin{array}{rcl} \text{for } a \in \mathscr{T}_i^1,\ \partial^c a = (K,L) \text{ then } (\varphi^\star_a, \varphi_a)_0 & = & (\operatorname{cotan} \theta_{a,K} + \operatorname{cotan} \theta_{a,L})/2 \\ \text{for } a \in \mathscr{T}_b^1,\ \partial^c a = (K) \text{ then } (\varphi^\star_a, \varphi_a)_0 & = & \operatorname{cotan} \theta_{a,K}/2\,. \end{array} \right. \tag{17}$$

The mixed Petrov-Galerkin discrete problem (13) for the Poisson equation has a unique solution and coincides with the classical "VF4" scheme introduced in [9] (see also Faille [8] and Eymard *et al.* [7]).

Theorem 1 has various consequences. Conditions in definition 1 that must be satisfied by Raviart Thomas dual basis are replaced by sufficient conditions on δ_K and g. In the sequel we will focus on such Raviart Thomas dual basis, though more general ones may exist: this will not be discussed in this contribution. Assuming the existence of g and δ_K, the construction of such dual basis is very delicate. No explicit representation can *a priori* be obtained. Nevertheless, a Raviart Thomas dual basis can be mathematically constructed by the following process. Consider $\varphi_{K,i} = \nabla u_{K,i}$ where $u_{K,i}$ is a solution of, $\Delta u_{K,i} = \delta_K \quad \text{on} \quad K, \quad \nabla u_{K,i} \cdot n = g_{K,i}$ on $a_{K,i}$ and $u_{K,i} \cdot n = 0$ elsewhere on ∂K. The compatibility condition for this problem is satisfied with the first statements and therefore $\varphi_{K,i}$ is well defined.
Whatever are δ_K and g, the coefficients $(\varphi^\star_a, \varphi_a)_0$ will be unchanged: they only depend on the mesh geometry and are given by eq. (17). Practically, this means that neither the $(\varphi^\star_a)_{a \in \mathscr{T}^1}$ nor δ_K and g need to be computed. The numerical scheme will always coincide with the "VF4" one. Eventually, this provides a new point of view for the understanding and analysis of finite volume methods.

5) Stability and convergence

General assumptions.

A couple of constant $0 < \theta_\star < \theta^\star < \pi/2$ is fixed and $\mathscr{T}$ will denote a mesh satisfying the uniform angle condition,

(18) $$\forall\, K \in \mathscr{T}^2, \quad i = 1,\ 2,\ 3: \quad \theta_\star \le \theta_{K,i} \le \theta^\star.$$

Theorem 1 implies that the mixed Petrov-Galerkin discrete problem (12) is independent on the particular choice made for the Raviart Thomas dual basis.

Theorem 2 [Error estimations]
There exists a constant C independent on $\mathscr{T}$ and of f in the Poisson problem so that the solution $(u_{\mathscr{T}}, p_{\mathscr{T}})$ of the mixed Petrov-Galerkin discrete problem (12) satisfies,

$$\|u_{\mathscr{T}}\|_0 + \|p_{\mathscr{T}}\|_{\mathrm{H(div,\Omega)}} \le C\, \|f\|_0 .$$

Denoting by u the exact solution to the Poisson problem and by $p = \nabla u$ the following error estimates moreover holds,

(19) $$\|u - u_{\mathscr{T}}\|_0 + \|p - p_{\mathscr{T}}\|_{\mathrm{H(div,\Omega)}} \le C h_{\mathscr{T}} \|f\|_1 ,$$

with $h_{\mathscr{T}}$ the mesh size.

Proof. We first prove that the mixed Petrov-Galerkin formulation has a unique solution depending continuously on the data thanks to Babuška's work [1]. The bilinear form B is continuous on V:

$$|B(\xi,\eta)| \ \le\ M\, \|\xi\|_V\, \|\eta\|_V\, , \forall \xi, \eta \in V .$$

The inf-sup stability condition relies on a stability result [3, 5, 6] and introduces a constant $\beta > 0$ such that for any mesh $\mathscr{T}$,

$$\forall \xi\ \in P^0 \times RT_0 \,\mathrm{such\,that}\ \|\ \xi\ \|_V = 1\, , \exists \eta\ \in\ P^0 \times RT_0^\star\, ,\ \|\ \eta\ \|_V \le\ 1\,\mathrm{and}\, B(\xi,\,\eta)\ \ge\ \beta\, .$$

The discrete "infinity condition" is satisfied [3]:

$$\forall \eta \in V^\star \backslash \{0\},\ \sup_{\xi \in V} B(\xi,\eta) = +\infty .$$

Then due to Babuška theorem valid also for Petrov-Galerkin mixed formulation the discrete scheme (12) has a unique solution and

$$\|\xi - \xi_{\mathscr{T}}\|_V\ \le\ \big(1 + \frac{M}{\beta}\big) \inf_{\zeta \in V_{\mathscr{T}}} \|\xi - \zeta\|_V\, .$$

In our case, this formulation is equivalent to

(20) $$\|u - u_{\mathscr{T}}\|_0 + \|p - p_{\mathscr{T}}\|_{\mathrm{div}} \le C \Big(\inf_{v \in P^0} \|u - v\|_0 + \inf_{q \in RT} \|p - q\|_{\mathrm{div}} \Big)$$

for a constant $C = 1 + \frac{M}{\beta}$ dependent of $\mathscr{T}$ only through the lowest and the highest angles θ_* and θ^*. We now precise an upper bound of the right-hand side of (20). With the interpolation operators $\Pi_0 : L^2(\Omega) \to P^0$ and $\Pi_{RT} : H^1(\Omega)^2 \to RT^0$, we have

$$\|\, u - u_{\mathscr{T}}\, \|_0 + \|\, p - p_{\mathscr{T}}\, \|_{\mathrm{div}} \le C \left(\|\, u - \Pi_0 u\, \|_0 + \|\, p - \Pi_{\mathrm{RT}} p\, \|_{\mathrm{div}} \right) .$$

On the other hand the interpolation errors are established by Raviart and Thomas [10] for the operator $\Pi_{\rm RT}$:

$$\| u - \Pi_0 u \|_0 \leq h_{\mathscr{T}} \| u \|_1 ,$$

$$\| p - \Pi_{\rm RT} p \|_0 \leq h_{\mathscr{T}} \| p \|_1 , \quad \| \operatorname{div} \big(p - \Pi_{\rm RT} p \big) \|_0 \leq h_{\mathscr{T}} \| \operatorname{div} p \|_1 .$$

Then

$$\| u - u_{\mathscr{T}} \|_0 + \| p - p_{\mathscr{T}} \|_{\rm div} \leq C h_{\mathscr{T}} \big(\| u \|_1 + \| p \|_1 + \| \operatorname{div} p \|_1 \big) .$$

Since $-\Delta u = f$ in Ω, with $f \in L^2(\Omega)$ and Ω convex, then $u \in H^2(\Omega)$ and $\|u\|_2 \leq c \|f\|_0$. Moreover $p = \nabla u$ and $\operatorname{div} p = -f$ leads to

$$\| u - u_{\mathscr{T}} \|_0 + \| p - p_{\mathscr{T}} \|_{\rm div} \leq C h_{\mathscr{T}} \big(2 \| f \|_0 + \| f \|_1 \big) .$$

Finally, we get

$$\| u - u_{\mathscr{T}} \|_0 + \| p - p_{\mathscr{T}} \|_{\rm div} \leq C h_{\mathscr{T}} \| f \|_1 ,$$

that is exactly (19). ■

6) Possible extensions

Our analysis for the Laplace equation is also *a priori* valid for three space dimensions. Moreover, the extension of the scheme to equations with tensorial coefficients is also possible in principle. To build a dual Raviart-Thomas basis for these problems is one of our objectives for a future contribution.

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